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SINGULAR STRESSES IN A SOFT FERROMAGNETIC ELASTIC SOLID WITH TWO COPLANAR GRIFFITH CRACKS

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Abstract—Following a linear theory for the soft ferromagnetic elastic materials, we consider the linear magnetoelastic problem for an infinite body with two coplanar Griffith cracks under the condition of plane strain. It is assumed that the soft ferromagnetic elastic solid is a homogeneous and isotropic one and is permeated by a uniform magnetostatic field normal to the cracks surfaces. By the use of Fourier transforms we reduce the problem to solving two simultaneous triple integral equations. These equations are exactly solved by using finite Hilbert transform techniques. The singular stresses near the crack tip are expressed in closed elementary forms and the influence of the magnetic fields upon the stress-intensity factors is shown graphically.

1. INTRODUCTION

In recent years, the theory of magnetoelasticity, which is concerned with the interacting effects of externally applied magnetic field on the elastic deformation of a solid body, has been developed rapidly because of the possibility of its extensive practical applications in various branches of science and technology. The interaction of stress, strain in an elastic body and magnetic fields can occur in a number of ways such as the effect of induced current and magnetization on the body and so on. Here we consider a soft ferromagnetic elastic solid. The first attempts of studying the effect of induced magnetization on the solid were made by Moon and Pao[1] in order to understand the magnetoelastic buckling of a beam-plate under a transverse magnetic induction. Later Pao and Yeh[2] gave the field equations and boundary conditions of the linear theory for a soft ferromagnetic elastic solids in a systematic way. Recently, based on the theory of Pao and Yeh the author has investigated the two and three-dimensional crack problems[3, 4] for the soft ferromagnetic elastic solid subjected to a uniform magnetostatic field normal to the cracks surfaces. These are, however, restricted to two-part mixed boundary value problems which are mathematically easy. Furthermore, their theory has not been applied to three-part mixed boundary value problems such as two coplanar Griffith cracks problem. Such problems are also of great practical importance.

In the present paper, we study the linear magnetoelastic problem for the soft ferromagnetic elastic solid with two coplanar Griffith cracks permeated by a uniform magnetostatic field normal to the cracks surfaces, which have innumerable applications in the field of fracture mechanics. By using Fourier transform techniques, the problem is reduced to that of solving two simultaneous triple integral equations. The two simultaneous triple integral equations are exactly solved by using finite Hilbert transform techniques. The singular stresses near the crack tip are then expressed in closed forms and the effect of magnetic fields upon the stress-intensity factors is shown graphically.

2. STATEMENT OF PROBLEM AND FUNDAMENTAL EQUATIONS

Let two open coplanar Griffith cracks be located in the interior of a homogeneous, isotropic, linearly elastic, soft ferromagnetic, infinite solid. Consider a rectangular cartesian coordinate system (x, y, z) such that these cracks are placed on the x-axis from $-b$ to $-a$ and from a to b as shown in Fig. 1. A uniform magnetic field is applied perpendicularly to the cracks surfaces. The external magnetic induction $\mathbf{B}_0$ is represented by $\mathbf{B}_0 = (0, B_{0y}, 0) = B_0\mathbf{e}_y$ where B_0 is constant, and $\mathbf{e}_y$ is a unit vector along y-axis.

For convenience, all magnetic quantities outside the solid will be denoted by the superscript

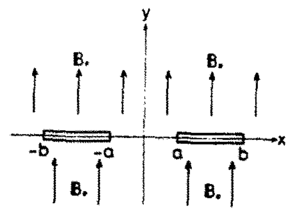

Fig. 1. Two coplanar Griffith cracks in a soft ferromagnetic elastic solid.

(e). The solutions for the rigid body state are

$$\begin{array}{lll} B_{0y}^{(e)} = B_0, & H_{0y}^{(e)} = B_0/\mu_0, & M_{0y}^{(e)} = 0 \\ B_{0y} = B_0, & H_{0y} = B_0/\mu_0\mu_r, & M_{0y} = \chi B_0/\mu_0\mu_r \end{array} \tag{1}$$

where H_{0y} and M_{0y} are the y-components of the magnetic intensity vector $\mathbf{H}_0$ and the magnetization vector $\mathbf{M}_0$, respectively, $\mu_0 = 4\pi\times10^{-7}$ newton/amper^2 (H/m) is the magnetic permeability of the vaccum, μ_r is the specific magnetic permeability and χ is the magnetic susceptibility.

We consider small perturbations of the magnetic intensity $\mathbf{h} = (h_x, h_y, 0)$, the magnetic induction $\mathbf{b} = (b_x, b_y, 0)$ and the magnetization $\mathbf{m} = (m_x, m_y, 0)$, which are characterized by a small displacement field $\mathbf{u} = [u_x(x, y), u_y(x, y), 0)]$ produced in the solid, and assume that all perturbations are independent of z. Hence the nontrivial components of the stresses and the magnetic quantities are[3]

$$\begin{aligned} t_{xx} &= 2\mu\left[\frac{\nu}{1-2\nu}(u_{x,x}+u_{y,y})+u_{x,x}\right] \\ t_{xy} &= t_{yx} = \mu(u_{x,y}+u_{y,x})+(B_0/\mu_r)m_x \\ t_{yy} &= 2\mu\left[\frac{\nu}{1-2\nu}(u_{x,x}+u_{y,y})+u_{y,y}\right]+2(B_0/\mu_r)m_y+\chi B_0^2/\mu_0\mu_r^2 \end{aligned} \tag{2}$$

$$\begin{aligned} \sigma_{Mxx} &= -\mu_0 H_0 h_y - \frac{1}{2}\mu_0 H_0^2 \\ \sigma_{Mxy} &= \sigma_{Myx} = \mu_0 B_0 h_x \\ \sigma_{Myy} &= \frac{1+2\chi}{\mu_r}B_0 h_y + \frac{1+2\chi}{2\mu_0\mu_r^2}B_0^2 \end{aligned} \tag{3}$$

$$\begin{array}{ll} h_x = m_x/\chi, & h_y = m_y/\chi \\ b_x = \mu_0\mu_r h_x, & b_y = \mu_0\mu_r h_y \end{array} \tag{4}$$

where t_{xx}, t_{xy} and t_{yy} are the magnetoelastic stresses, σ_{Mxx}, σ_{Mxy} and σ_{Myy} are the Maxwell stresses, the Lamé constants λ and μ are replaced by ν and μ with $\lambda = 2\mu\nu/(1-2\nu)$, ν is the Poisson's ratio, and a comma denotes partial differentiation with respect to the coordinate. In this case the linearized field equations can be also expressed in the following forms:

$$\begin{gathered} \nabla^2 u_x + \frac{1}{1-2\nu}(u_{x,x}+u_{y,y})_{,x} + \frac{2\chi B_0}{\mu\mu_r}h_{x,y} = 0 \\ \nabla^2 u_y + \frac{1}{1-2\nu}(u_{x,x}+u_{y,y})_{,y} + \frac{2\chi B_0}{\mu\mu_r}h_{y,y} = 0 \\ \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \end{gathered} \tag{5}$$

$$h_x = \phi_{,x}, \qquad h_y = \phi_{,y}, \qquad \nabla^2\phi = 0 \tag{6}$$

$$h_x^{(e)} = \phi_{,x}^{(e)}, \qquad h_y^{(e)} = \phi_{,y}^{(e)}, \qquad \nabla^2\phi^{(e)} = 0 \tag{7}$$

where ϕ is the magnetic potential. The correspondence mixed boundary conditions, as derived from[3], are that at $y = 0$,

$$h_x^{(e)}(x, 0) - h_x(x, 0) = -\frac{\chi B_0}{\mu_0 \mu_r} u_{y,x}(x, 0); \qquad a < |x| < b$$

$$\phi(x, 0) = 0; \qquad |x| \leq a, \qquad b \leq |x|$$

$$b_y^{(e)}(x, 0) - b_y(x, 0) = 0; \qquad a < |x| < b$$

$$\phi^{(e)}(x, 0) = 0; \qquad a < |x| < b \tag{8}$$

$$t_{yx}(x, 0) = 0; \qquad |x| < \infty$$

$$t_{yy} = \frac{\chi^2}{\mu_r}\left\{B_0 h_y(x, 0) + \frac{B_0^2}{2\mu_0 \mu_r}\right\}; \qquad a < |x| < b \tag{9}$$

$$u_y(x, 0) = 0; \qquad |x| \leq a, \qquad b \leq |x|$$

3. ANALYSIS

The solutions of eqns (5)–(7) for $y \geq 0$ will be of the following forms in terms of the unknown functions $A(\alpha)$, $B(\alpha)$, $a(\alpha)$ and $a_e(\alpha)$,

$$u_x = \frac{2}{\pi}\int_0^\infty \left[A(\alpha) - (3 - 4\nu)\frac{B(\alpha)}{\alpha} + B(\alpha)y + 2(1 - 2\nu)(\chi B_0/\mu\mu_r)a(\alpha)\right] e^{-\alpha y} \sin \alpha x \, d\alpha$$

$$u_y = \frac{2}{\pi}\int_0^\infty \{A(\alpha) + B(\alpha)y\} e^{-\alpha y} \cos\alpha x \, d\alpha \tag{10}$$

$$\phi = -\frac{2}{\pi}\int_0^\infty \alpha a(\alpha) e^{-\alpha y} \cos\alpha x \, d\alpha$$

$$\phi^{(e)} = \frac{2}{\pi}\int_0^\infty a_e(\alpha) sh\alpha y \cos\alpha x \, d\alpha. \tag{11}$$

Making use of mixed boundary conditions (8) and (9), we have the two simultaneous triple integral equations:

$$\int_0^\infty \alpha\{a(\alpha) - (\chi B_0/\mu_0\mu_r)A(\alpha)\} \sin\alpha x \, d\alpha = 0; \qquad a < |x| < b$$

$$\int_0^\infty a(\alpha) \cos\alpha x \, d\alpha = 0; \qquad |x| \leq a, \qquad b \leq |x| \tag{12}$$

$$\int_0^\infty \alpha[A(\alpha) + (\chi B_0/2\mu\mu_r)\{1 - 2\nu - 2(1 - \nu)\chi\}a(\alpha)] \cos\alpha x \, d\alpha = \frac{\pi}{2\mu}(1 - \nu)P_{h0}; \qquad a < |x| < b$$

$$\int_0^\infty A(\alpha) \cos\alpha x \, d\alpha = 0; \qquad |x| \leq a, \qquad b \leq |x| \tag{13}$$

in which the unknown $A(\alpha)$ is related to $B(\alpha)$ and $a(\alpha)$ as follows:

$$\alpha A(\alpha) = 2(1 - \nu)B(\alpha) - (3 - 4\nu)(\chi B_0/2\mu\mu_r)\alpha a(\alpha). \tag{14}$$

Adopted in the first equation of (13) are the contractions

$$P_{h0} = -\mu\chi(\chi - 2)b_c^2/2\mu_r^2, \qquad b_c^2 = B_0^2/\mu_0\mu. \tag{15}$$

To solve the foregoing set of simultaneous triple integral equations, the method of Lowengrub and Srivastava[5] can be used with slight modification. If we make the integral representations for $a(\alpha)$ and $A(\alpha)$

$$a(\alpha)=\frac{1}{\alpha}\int_a^b h(\xi)\sin\alpha\xi\,d\xi$$
$$A(\alpha)=\frac{1}{\alpha}\int_a^b \psi(\xi)\sin\alpha\xi\,d\xi. \tag{16}$$

If we now substitute the eqns (16) into the first equation of (12), after some manipulations, we have

$$h(\xi)-(\chi B_0/\mu_0\mu_r)\psi(\xi)=0. \tag{17}$$

From the first equation of (13), we also have the following integral equation

$$\int_a^b [\psi(\xi)+\{1-2\nu-2(1-\nu)\chi\}(\chi B_0/2\mu\mu_r)h(\xi)]\frac{\xi}{\xi^2-x^2}d\xi=\frac{\pi P_{h0}}{2\mu}. \tag{18}$$

From the second equations of (12) and (13) and the definitions (16) it is clear that the integral equation must be solved under the following single-valuedness conditions:

$$\int_a^b h(\xi)\,d\xi=0$$
$$\int_a^b \psi(\xi)\,d\xi=0. \tag{19}$$

Using the theorem for finite Hilbert transform, we find that the solution to the integral equation (18) is given by

$$\psi(\xi)+\{1-2\nu+2(1-\nu)\chi\}(\chi B_0/2\mu\mu_r)h(\xi)=-\frac{2}{\pi\mu}P_{h0}\left(\frac{\xi^2-a^2}{b^2-\xi^2}\right)^{1/2}\int_a^b\left(\frac{b^2-y^2}{y^2-a^2}\right)^{1/2}\frac{y}{y^2-\xi^2}dy$$
$$+\frac{2\mu_r^2+\{1-2\nu-2(1-\nu)\chi\}(\chi b_c)^2}{2\mu_r^2}C\{(\xi^2-a^2)(b^2-\xi^2)\}^{-1/2} \tag{20}$$

where C is an arbitrary constant. By substituting $\psi(\xi)$, which is obtained from the eqns (17) and (20), into the second condition of (19), we can easily derive

$$C=\frac{2(1-\nu)\mu_r^2b^2P_{h0}}{\mu[2\mu_r^2+\{1-2\nu-2(1-\nu)\chi\}(\chi b_c)^2]}\{(a/b)^2-E/F\}. \tag{21}$$

Here, $F=F\{\pi/2,(b^2-a^2)^{1/2}/b\}$ and $E=E\{\pi/2,(b^2-a^2)^{1/2}/b\}$ are the complete elliptic integrals of the first and second kind, respectively. The functions $a(\alpha)$ and $A(\alpha)$ can now be determined from the relations (16).

By superposing the magnetoelastic stresses t_{ij} and the Maxwell stresses σ_{Mij}, the complete stresses t_{cij} are obtained in the form:

$$t_{cij}=t_{ij}+\sigma_{Mij}. \tag{22}$$

Considering the behavior of $\alpha\to\infty$ in the expressions of stresses, we obtain the terms which will contribute the singularity of stresses as follows;

$$t_{xx}\sim-\frac{\mu}{2\pi(1-\nu)\mu_r^2y_0}\int_a^b\left[y_1\frac{\xi-x}{y^2+(\xi-x)^2}-2y_2\frac{y^2(\xi-x)}{\{y^2+(\xi-x)^2\}^2}\right]\psi(\xi)\,d\xi$$
$$t_{xy}\sim-\frac{\mu y_3 y}{2\pi(1-\nu)\mu_r^2y_0}\int_a^b\frac{y^2-(\xi-x)^2}{\{y^2+(\xi-x)^2\}^2}\psi(\xi)\,d\xi \tag{23}$$

$$t_{yy} \sim -\frac{\mu}{2\pi(1-\nu)\mu_r^2 y_0}\int_a^b \left[y_1\frac{\xi-x}{y^2+(\xi-x)^2}+2y_2\frac{y^2(\xi-x)}{\{y^2+(\xi-x)^2\}^2}\right]\psi(\xi)\,d\xi$$

$$\begin{aligned}
\sigma_{Mxx} &\sim \frac{\mu\chi b_c^2}{\pi\mu_r^2}\int_a^b \frac{\xi-x}{y^2+(\xi-x)^2}\psi(\xi)\,d\xi \\
\sigma_{Mxy} &\sim -\frac{\mu\chi b_c^2}{\pi\mu_r}\int_a^b \frac{y}{y^2+(\xi-x)^2}\psi(\xi)\,d\xi \\
\sigma_{Myy} &\sim -\frac{\mu\chi(1+\mu_r)b_c^2}{\pi\mu_r^2}\int_a^b \frac{\xi-x}{y^2+(\xi-x)^2}\psi(\xi)\,d\xi
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
y_1 &= \{2\mu_r^2+(1-2\nu)(\chi b_c)^2\}y_0 \\
y_2 &= \{2\mu_r^2+(3-4\nu)(\chi b_c)^2\}y_0 \\
y_3 &= \{2\mu_r^2+(3-4\nu)(\chi b_c)^2\}y_0 \\
y_0 &= 1/[2\mu_r^2+\{2(1-\nu)+(5-6\nu)\chi\}(\chi b_c^2).
\end{aligned} \tag{25}$$

In order to find these stresses in closed forms, we use the polar coordinates defined as

$$\begin{aligned}
\rho_a &= \{(x-a)^2+y^2\}^{1/2}, \qquad \theta_a = \tan^{-1}\{y/(x-a)\} \\
\rho_b &= \{(x-b)^2+y^2\}^{1/2}, \qquad \theta_b = \tan^{-1}\{y/(x-b)\}
\end{aligned} \tag{26}$$

and put $\psi(\xi)$, with consideration of its singularity, as follows

$$\psi(\xi) = \Psi(\xi)/\{(\xi-a)(b-\xi)\}^{1/2}. \tag{27}$$

Then the singular parts of t_{xx}, t_{xy}, t_{yy} and σ_{Mxx}, σ_{Mxy}, σ_{Myy} are obtained by using the theorem [6] on the behavior of Cauchy integral near the ends of the path of integration as follows

$$\begin{aligned}
t_{xx} &\sim \{y_1+y_2\cos(\theta_a/2)\cos(3\theta_a/2)\}\sin(\theta_a/2)\frac{K_{h1a}}{(2\rho_a)^{1/2}}+\{y_1-y_2\sin(\theta_b/2)\sin(3\theta_b/2)\}\cos(\theta_b/2)\frac{K_{h1b}}{(2\rho_b)^{1/2}} \\
t_{xy} &\sim \frac{y_3}{2}\left[\sin\theta_a\sin(3\theta_a/2)\frac{K_{h1a}}{(2\rho_a)^{1/2}}+\sin\theta_b\cos(3\theta_b/2)\frac{K_{h1b}}{(2\rho_b)^{1/2}}\right] \\
t_{yy} &\sim \{y_1-y_2\cos(\theta_a/2)\cos(3\theta_a/2)\}\sin(\theta_a/2)\frac{K_{h1a}}{(2\rho_a)^{1/2}}+\{y_1+y_2\sin(\theta_b/2)\sin(3\theta_b/2)\}\cos(\theta_b/2)\frac{K_{h1b}}{(2\rho_b)^{1/2}}
\end{aligned} \tag{28}$$

$$\begin{aligned}
\sigma_{Mxx} &\sim -2(1-\nu)y_0\chi b_c^2\left[\sin(\theta_a/2)\frac{K_{h1a}}{(2\rho_a)^{1/2}}+\cos(\theta_b/2)\frac{K_{h1b}}{(2\rho_b)^{1/2}}\right] \\
\sigma_{Mxy} &\sim 2(1-\nu)y_0\chi\mu_r b_c^2\left[\cos(\theta_a/2)\frac{K_{h1a}}{(2\rho_a)^{1/2}}-\sin(\theta_b/2)\frac{K_{h1b}}{(2\rho_b)^{1/2}}\right] \\
\sigma_{Myy} &\sim 2(1-\nu)y_0\chi(1+\mu_r)b_c^2\left[\sin(\theta_a/2)\frac{K_{h1a}}{(2\rho_a)^{1/2}}+\cos(\theta_b/2)\frac{K_{h1b}}{(2\rho_b)^{1/2}}\right]
\end{aligned} \tag{29}$$

where K_{h1a} and K_{h1b} are the stress-intensity factors at the inner and the outer tips of the crack, respectively, and are defined by the following equations:

$$\begin{aligned}
K_{h1a} &= \lim_{x\to a^-}\{2(a-x)\}^{1/2}t_{cyy|y=0} = \frac{P_{h0}b^{1/2}}{[2\mu_r^2+\{1-2\nu-2(1-\nu)\chi\}(\chi b_c)^2]y_0}\left[\frac{E/F-(a/b)^2}{(a/b)^{1/2}\{1-(a/b)^2\}^{1/2}}\right] \\
K_{h1b} &= \lim_{x\to b^+}\{2(b-x)\}^{1/2}t_{cyy|y=0} = \frac{P_{h0}b^{1/2}}{[2\mu_r^2+\{1-2\nu-2(1-\nu)\chi\}(\chi b_c)^2]y_0}\left[\frac{1-E/F}{\{1-(a/b)^2\}^{1/2}}\right].
\end{aligned} \tag{30}$$

We might also note that when $a/b = 0$, the cracks merge into a single one of width $2b$. In this case, the second equation of (30) reduces to the simple expression

$$K_{h1b|a/b=0} = \frac{P_{h0}b^{1/2}}{[2\mu_r^2 + \{1 - 2\nu - 2(1-\nu)\chi\}(\chi b_c)^2]y_0} \tag{31}$$

obtained by the author[3] for the stress-intensity factor for a single crack. The critical magnetic induction B_{0cr} at which the surfaces of the cracks are unstable is

$$B_{0cr} = \left[\frac{2\mu\mu_0\mu_r^2}{\chi^2\{-1+2\nu+2(1-\nu)\chi\}}\right]^{1/2} \tag{32}$$

which agrees with the result for a single crack[3, 4].

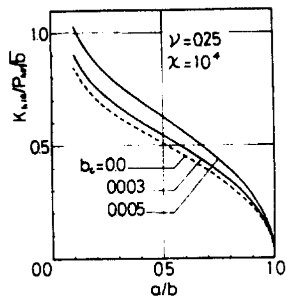

Fig. 2. Effect of magnetic fields on the stress-intensity factors at the inner tip of the crack with a/b.

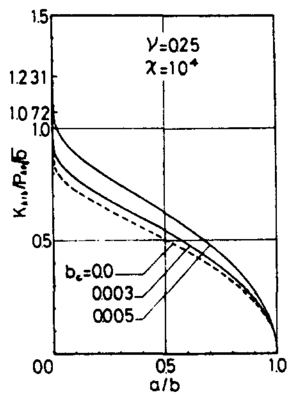

Fig. 3. Effect of magnetic fields on the stress-intensity factors at the outer tip of the crack with a/b.

4. NUMERICAL RESULTS

To examine the effect of magnetic fields on the stress-intensity factors, the normalized stress-intensity factors have been calculated for any prescribed values of Poisson's ratio ν and magnetic susceptibility χ. Figure 2 shows the variations of $K_{h1a}/P_{h0}b^{1/2}$ at the inner tip of the crack with a/b for $b_c = 0$, 0.003, 0.005, $\nu = 0.25$ and $\chi = 10^4$. The same kind of results for $K_{h1b}/P_{h0}b^{1/2}$ at the outer tip of the crack is shown in Fig. 3. The dashed curves obtained for the case $b_c = 0$ coincide with the purely elastic case. The existence of the magnetic field produces higher stress concentration in the neighborhood of the crack tip. $K_{h1b}/P_{h0}b^{1/2}$ tends to the result for the single crack by the author[3] as $a/b \to 0$ and tends to zero as $a/b \to 1$. Numerical results for $K_{h1a}/P_{h0}b^{1/2}$ and $K_{h1b}/P_{h0}b^{1/2}$ show that $K_{h1a}/P_{h0}b^{1/2}$ is always larger than $K_{h1b}/P_{h0}b^{1/2}$. It follows from this that the form of the two coplanar Griffith cracks is unstable. The development of the cracks for monotonous increase of the load P_{h0} starts at points of the inner tips and the two cracks transform into a single crack of width $2b$.

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